MEGL - Braess Paradox

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INTRODUCTION

- **Network Synchronization**
  - Many systems (power grids, traffic, neurons) need their components **to move together (synchronize).**
  - We model this using **the Kuramoto Model** to assess a **network of nodes.**
  - Whether the system synchronizes depends on the **coupling strength**.
- **The Problem**
  - Adding a new connection (**an edge)** should help a network synchronize, but sometimes it makes things worse.
  - The network then **needs stronger coupling** to stay aligned.
  - This counter-intuitive failure is **Braess's Paradox.**

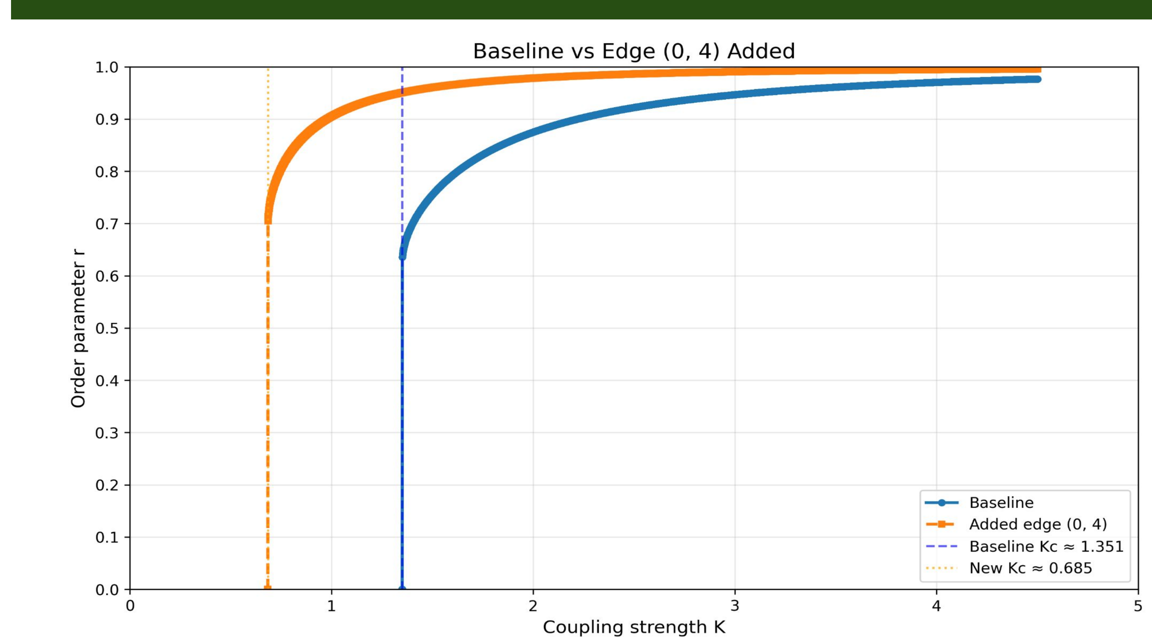
Our Goal: Can we develop an analytical expression that predicts which added edges will cause Braess's paradox in oscillator networks?

The Kuramoto Model

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^{N} W_{ij} \sin(\theta_j - \theta_i)$$

- $\theta_i \longrightarrow$ The phase of oscillator i
- $\omega_i \longrightarrow$ Natural frequency of i
- $K \longrightarrow$ Coupling strength constant
- $W_{ij} \longrightarrow$ Adjacency matrix between i and j
- $\sin(\theta_j - \theta_i) \longrightarrow$ Interaction between i and j pulling phases
- $\sum_j W_{ij} \sin(\theta_j - \theta_i) \longrightarrow$ Total influence of all of i's neighboring nodes

Braess Paradox



This Graph Depicts the change in Coupling Strength and the jump in Kc value that could occur.

METHODOLOGY

Implicit Function Theorem (IFT)

- IFT tells us how the synchronized solution changes when we change the network.
  - Like adding an edge.
- It requires a full rank Jacobian

Projection & Cokernel

$$DG = \begin{bmatrix} D_\theta F & 0 & F_K & F_a \\ D_\theta(DF\mathbf{v}) & DF & (DF)_K\mathbf{v} & (DF)_a\mathbf{v} \\ \vdots & \textit{constraints} & \vdots & \vdots \end{bmatrix}$$

$$\text{Cokernel vectors:} \quad \psi_1 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \quad \psi_2 = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} \quad \Longrightarrow \quad \psi_i^T DG = 0.$$

- The cokernel of DG is two-dimensional, spanned by **ψ1** and **ψ2**.
- They represent the directions where the system loses rank at the critical coupling.
- Projection onto **ψ1**and **ψ2** removes those unsolvable direction.
- This restores the right dimensions so we can apply IFT.

Augmented System (G)

$$G(\theta, \mathbf{v}, K_c, a) = \begin{bmatrix} F(\theta, K_c) \\ J\,\mathbf{v} \\ \mathbf{1}^T\mathbf{v} \\ \theta_1 \\ \mathbf{v}^T\mathbf{v} - 1 \end{bmatrix} = 0.$$

- Each row of G enforces a condition:
  - **F(θ, Kc) = 0** →Steady state equation (when our system is in sync).
  - **Jv = 0** → Critical point where our system loses stability (eigenvalue of **v** = 0).
  - $\mathbf{1^Tv = 0}$ → Orthogonality for v to the trivial direction.
  - **θ1 = 0** → Gauge fix for rotational symmetry (used as a reference point).
  - **vTv -1 = 0** → Normalizes **eigenvector v** to remove scaling freedom.
- Our system, however, does not satisfy the dimensions required for IFT ($R^{(2n+2)} \rightarrow R^{(2n+3)}$).

Deriving K′(0)

Linearize: $F_\theta\,\theta'(0) + F_K\,K'(0) + F_a = 0$

→ Project with $\mathbf{v}^T$: $DF(\theta_0)\mathbf{v} = 0 \Rightarrow \mathbf{v}^T F_\theta = 0$

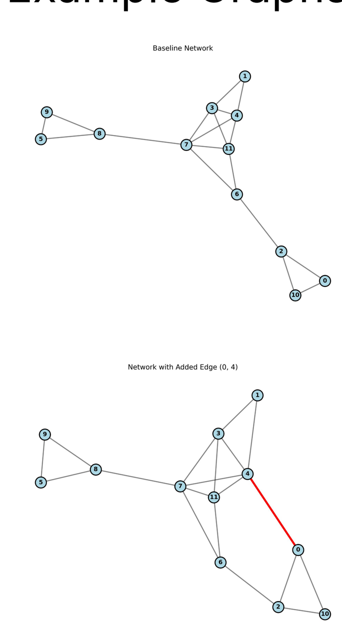
→ Reduced Equation: $\mathbf{v}^T F_K\,K'(0) + \mathbf{v}^T F_a = 0$

→ Final Expression: $K'(0) = -\frac{\mathbf{v}^T F_a}{\mathbf{v}^T F_K}$

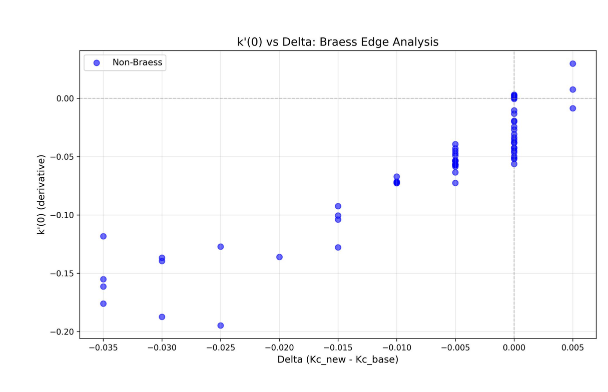
→ Interpretation: $K'(0) > 0 \Rightarrow$ Braess, $K'(0) < 0 \Rightarrow$ Helpful edge

Simulation

Example Graphs

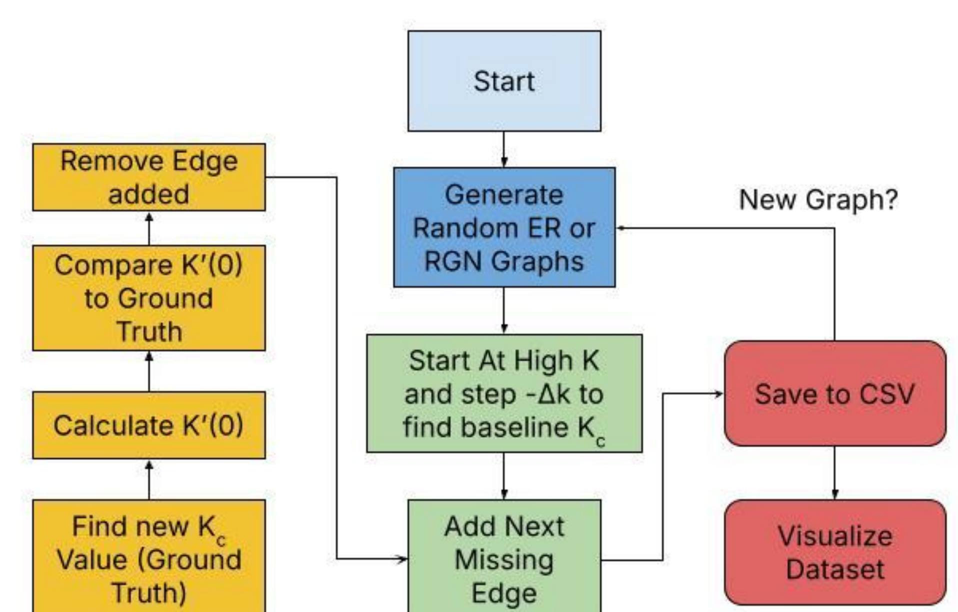


Ideal graph

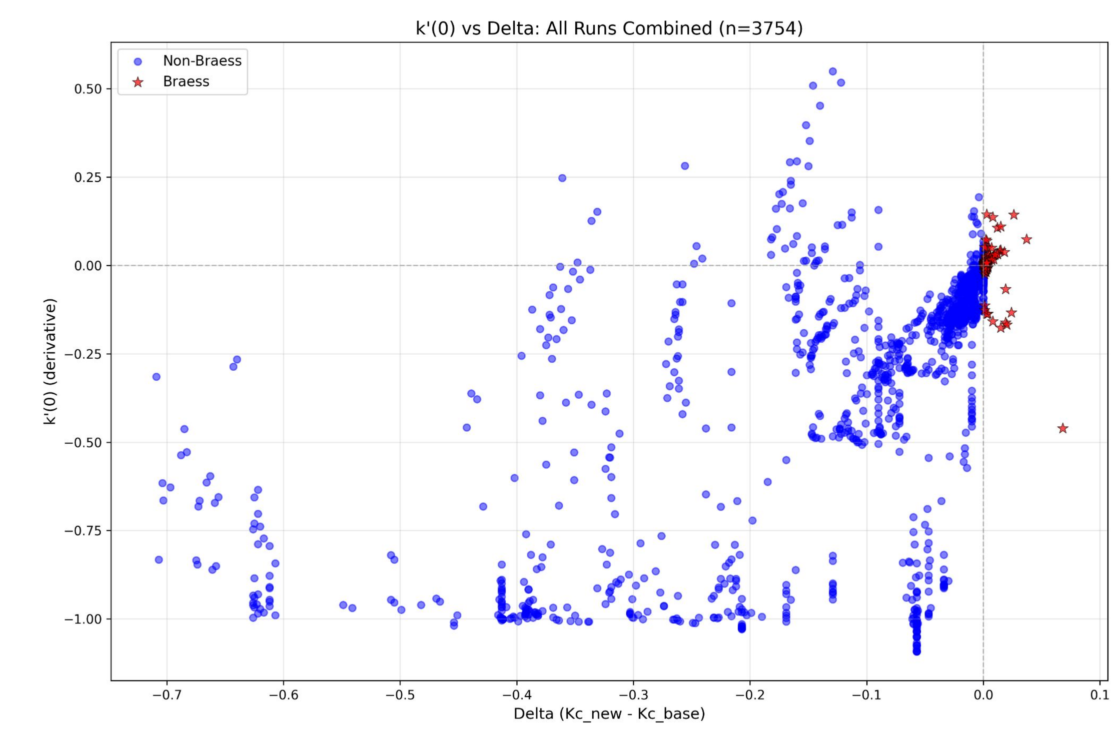


Our Simulation: 100+ graphs, 10,000+ data points, mix of ER & RGN Graphs

Code Logic



RESULTS



Example Kc Shift Graph over 10,000+ Edges

Confusion Matrix	*True Braess Edge*	*True Non-Braess Edge*
Predicted Braess	500	26
Predicted Non Braess	1300	6432

Metric	*Overall Accuracy*
Our Analytical Method	80%
Previous	N/A

Next Steps & Disucssion

Next Steps

- Validate analytical method on broader network classes
- Test on larger and more realistic networks
- Investigate high false-positive rate

Future Work

- Enhance explainability of predictions
- Identify structural causes behind false positives
- Develop theory linking network structure to Braess behavior?

Citations

https://iopscience.iop.org/article/10.1088/1367-2630/14/8/083036

https://journals.aps.org/pre/abstract/10.1103/PhysRevE.93.032222

https://www.nature.com/articles/s41467-022-32917-6